

Beamforming algorithms - beamformers

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Abstract—Beamforming is the name given to a wide variety of array processing algorithms that focus or steer the array in a particular direction. Beamforming techniques are used to enhance directivity, and to aim the focus of the array without having to change it physically. To produce a common output the signals from the individual sensors are combined according to a certain algorithm.

Index Terms—Beamforming, delay-and-sum, minimum variance

I. DELAY-AND-SUM BEAMFORMING

ONE of the most common and robust beamforming algorithms is the conventional beamformer, also known as the Bartlett beamformer, or delay-and-sum (DAS) beamforming. The DAS beamformer applies a delay and an amplitude weight to the output of each sensor, and then sums the resulting signals. The delays are chosen to maximize the array's sensitivity to incoming waves from a particular direction. By adjusting the delays, the array's look-direction can be steered towards the source, and the waveforms captured by the individual sensors add constructively. This means that signals at particular angles experience constructive interference, while others experience destructive interference.

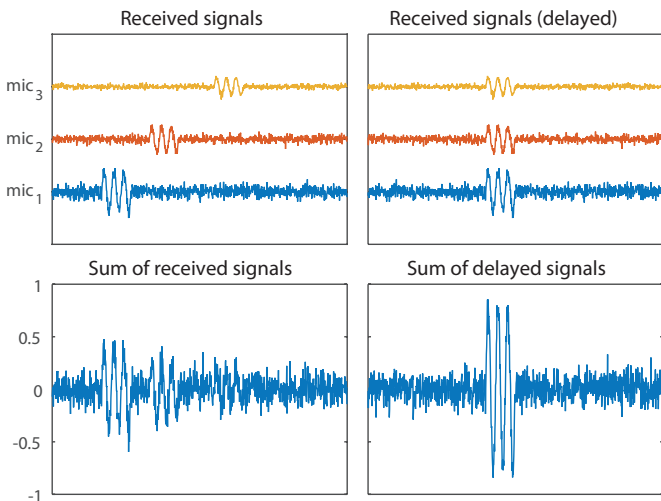


Fig. 1. The wavefield is arriving at individual microphones located at different positions at different times. Summing the signals directly distorts the signal, whereas by delaying the signals the waveforms captured by the individual sensors add constructively.

Consider an array consisting of M sensors that are located at different positions in space $\vec{x}_m = [x_m, y_m, z_m]$ that measures a wavefront $f(\vec{x}, t)$. The waveform spatially sampled by

the m th sensor is $y_m(t) = f(\vec{x}_m, t)$. The DAS beamformer consists of applying a delay Δ_m and an amplitude weight w_m to the output of each sensor, and then summing the resulting signals as displayed in Fig. 2.

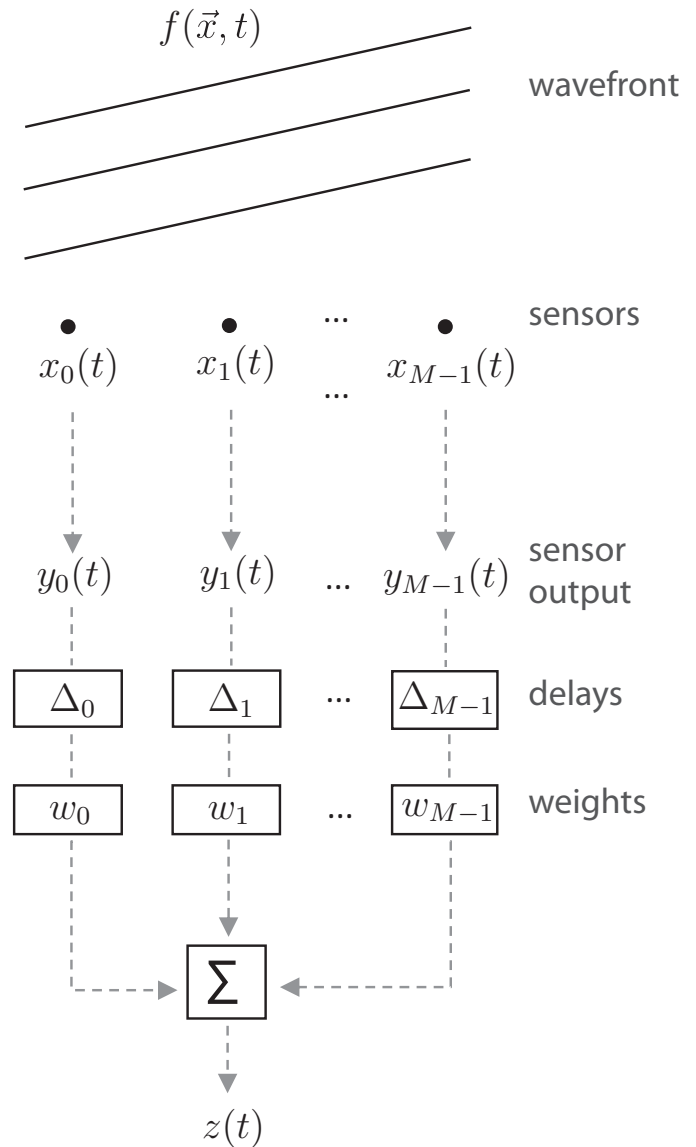


Fig. 2. Delay-and-sum (DAS) beamforming, also known as the conventional (or Bartlett) beamformer

The delays are chosen to maximize the array's sensitivity to waves propagating from a particular direction. By adjusting the delays, the array's direction of look can be steered towards the source, and the waveforms captured by the individual sensors add constructively. This operation is sometimes called *stacking*. Weighting the different sensors of the array differently may be seen as a gain factor for the individual

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sensors, and enhances the shape and reduces sidelobe levels of the listening beam. As opposed to adaptive methods, the sensor weights for the DAS beamformer are chosen in advance and independently of the received waveform. The DAS beamformer's output in the time domain is then

$$z(t) = \sum_{m=0}^{M-1} w_m \cdot y_m(t - \Delta_m) \quad (1)$$

The basic idea in beamforming for is then to use the set of delays to steer the array to different directions or points in a scanning plane. When the steering direction coincides with a source, the maximum output power will be observed. By interpolating the measured output power from all the scanning points, it is possible to colour the spatial power (power across the scanning plane) and make an acoustic image.

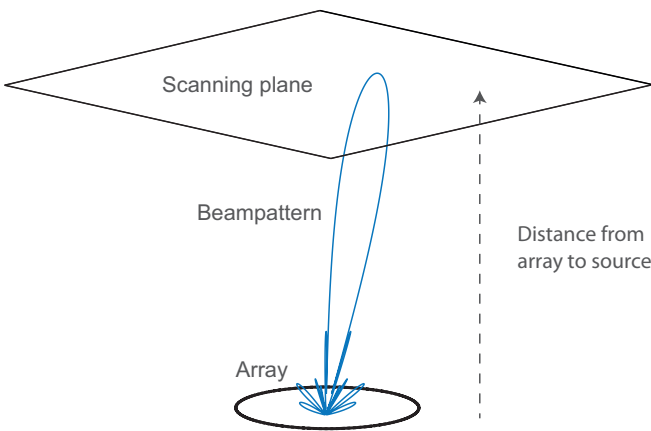


Fig. 3. The basic idea behind acoustic camera is to steer the listening direction of the array on different points in a scanning plane, measure the power from each point, and interpolate the values to create an image.

Defining the set of listening points in the scanning plane as $\vec{x}_s = [x_s, y_s, z_s]$, the set of delays Δ_m to steer the beam to a specific point are then calculated as

$$\Delta_m = \frac{|\vec{x}_s - \vec{x}_m|}{c} = \frac{\sqrt{(x_s - x_m)^2 + (y_s - y_m)^2 + (z_s - z_m)^2}}{c} \quad (2)$$

where c is the speed of sound. Remember that the DAS equation given in (1) is for a single point only, so calculation of time delays, delaying signals and summing of signals from all the different sensors has to be done for all the scanning points.

II. ARRAY OUTPUT FOR DELAY-AND-SUM BEAMFORMER

Now say we want to characterize the array sensitivity to a single frequency wave from an arbitrary incidence angle when using the DAS beamformer. That is we want to characterize the array itself when scanning over all incidence angles rather than only points in a plane. First consider the input to a single sensor as

$$y_m(t) = e^{j(\omega t - \vec{k} \cdot \vec{x}_m)} \quad (3)$$

where $\omega = 2\pi f$ is the frequency of the input signal with frequency f . The wavenumber vector (or wave vector) \vec{k} is the propagation vector giving both the magnitude and direction of arrival of the incident plane wave. As before \vec{x}_m is the position in space of the receiving sensor. By using the same input signal as in (3), the delayed signal may now be stated in terms of a phase shift rather than time delay as

$$y_m(t - \Delta_m) = y_m(t) \cdot e^{-j\omega\Delta_m} \quad (4)$$

Remember now that the signal $y_m(t)$ is the received signal from individual sensors and will be different for different sensors, as seen on the top left of Fig. 1, and $e^{-j\omega\Delta_m}$ represents the phase delay associated with the signal at the m th microphone. The DAS beamformer output may again be stated as in (1)

$$z(t) = \sum_{m=0}^{M-1} w_m \cdot y_m(t) \cdot e^{-j\omega\Delta_m} \quad (5)$$

If we now include these phase delays in the received signal vector $\mathbf{Y} = y_m(t) \cdot e^{-j\omega\Delta_m}$, we may write (5) in vector notation as

$$z = \mathbf{w}^H \mathbf{Y} \quad (6)$$

where \mathbf{Y} is the vector of the received signal from each sensor with its associated phase delay, \mathbf{w} is the weighting vector and H denotes the complex conjugate transpose. By using the vector notation given in (6), and assuming we have already steered the array to the desired direction, we can calculate the power, or the variance, of the output signal as

$$P(z) = \sigma^2 = \mathbb{E}\{|z|^2\} = \mathbf{w}^H \mathbf{R} \mathbf{w} \quad (7)$$

where $\mathbf{R} = \mathbb{E}\{\mathbf{Y}\mathbf{Y}^H\}$ is the correlation matrix of the incoming signal.

In (5) the phase delays associated with each individual sensor $e^{-j\omega\Delta_m}$ is the so called steering vector \mathbf{e} , and governs how we want to steer the beam of our array. Now suppose we want to measure the output power as a function of scanning angle or rather as a function of steering vector. This is termed the *steered response* and is the power of the beamformer output in the frequency domain. This array output power spectral density may then be expressed by using the correlation matrix \mathbf{R} and the steering vector \mathbf{e} as

$$P(\mathbf{e}) = \mathbf{e}^H \mathbf{R} \mathbf{e} \quad (8)$$

In essence, to calculate the spatial spectrum of the DAS beamformer for a specific array, steer the array to the desired direction and use (7) to calculate the output power. Or a different and equal approach, is to weight the signals on input, and use (8) to calculate the power for an arbitrary scanning angle.



III. MINIMUM VARIANCE BEAMFORMING

For the DAS beamformer the weighting of elements is predefined and stays the same regardless of input. A different approach would be to change the weighting of elements based on the input signals, or better yet, to *adapt* the weighting of elements to the input. A different algorithm that uses such an approach is the so called minimum variance distortionless response (MVDR) algorithm, or minimum variance (MV) for short. The basic idea, and the basis for the name, is to minimize the power or variance $P(z)$ of the output signal $z(t)$, all while the desired signal in our listening direction is not distorted. That means we want to force the beampattern of our array to have unity gain in our listening direction, while we minimize the impact from all other sources.

$$\begin{aligned} & \min P(z) \\ & \text{subject to } \mathbf{w}^H \mathbf{e} = 1 \end{aligned} \quad (9)$$

The solution for optimum weights to the above restrictions is given as

$$\mathbf{w} = \frac{\mathbf{R}^{-1} \mathbf{e}}{\mathbf{e}^H \mathbf{R}^{-1} \mathbf{e}} \quad (10)$$

The optimum weight vector now depends both on the input signals given by the spatial correlation matrix, and also on the steering vector which gives the angle of the listening direction of the array. As various directions are scanned, the optimal weights will change and adapt to the signals and noise in the observations.

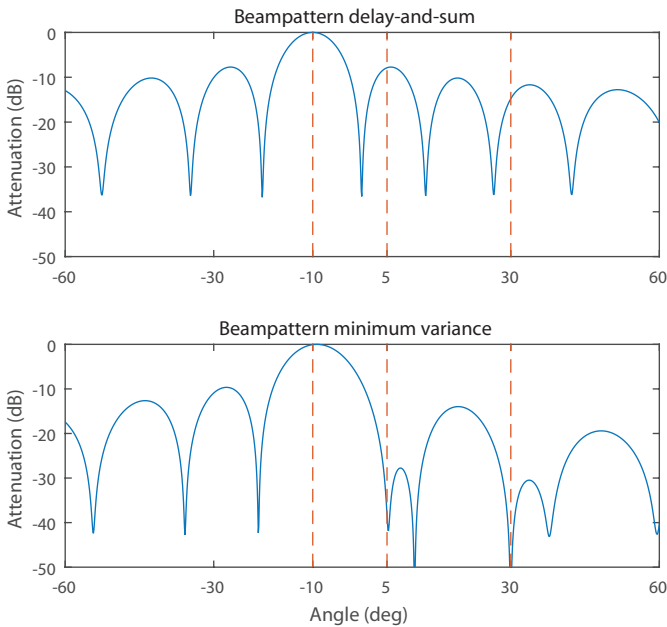


Fig. 4. Beampattern of DAS and MV when steered to -10 degrees. The input signal consists of three sources arriving at incidence angles -10, 5 and 30 degrees

In Fig. 4 we have three input signals arriving at -10, 5 and 30 degrees respectively, with the array being steered to the incidence angle of the first source, so the array will

have its mainlobe pointed in this direction. By looking at the beampattern of the DAS beamformer shown on top in Fig. 4, it is clear how the obtained signal will be distorted by signals arriving at an incidence angle that corresponds to the location of one of the side lobes of the array. Now focusing on the MV beampattern on the bottom, we see how the beampattern is forced to have minimum energy at arriving angles corresponding to other sources. This is what makes the MV algorithm so great, we can diminish the impact of interfering sources while still having maximum energy in our listening direction.

The optimal weights in (10) will give the corresponding spatial spectrum of the minimum variance beamformer as

$$P(\mathbf{e}) = \frac{1}{\mathbf{e}^H \mathbf{R}^{-1} \mathbf{e}} \quad (11)$$

By using the same input signals as in Fig. 4, we can calculate the steered response for both the DAS and MV algorithm as seen in Fig. 5. Clearly the MV algorithm has a strong increase in resolution over the DAS beamformer.

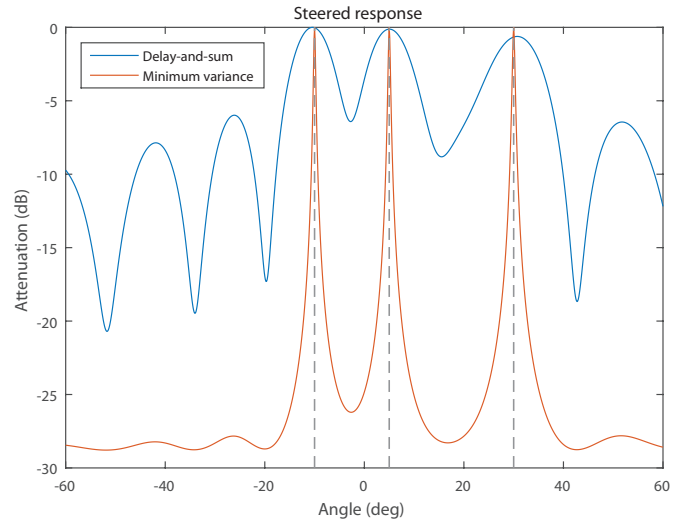


Fig. 5. Steered response of the DAS and MV algorithm. The input signal consists of three sources arriving at incidence angles -10, 5 and 30 degrees

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APPENDIX

Say we want to characterize the array sensitivity to a single frequency wave from an arbitrary incidence angle when using the delay-and-sum (DAS) beamformer. The incidence angle in spherical coordinates is then given as the elevation θ , which is the normal incidence angle, and azimuth ϕ which is the angle in the XY plane as illustrated in Fig. 6.

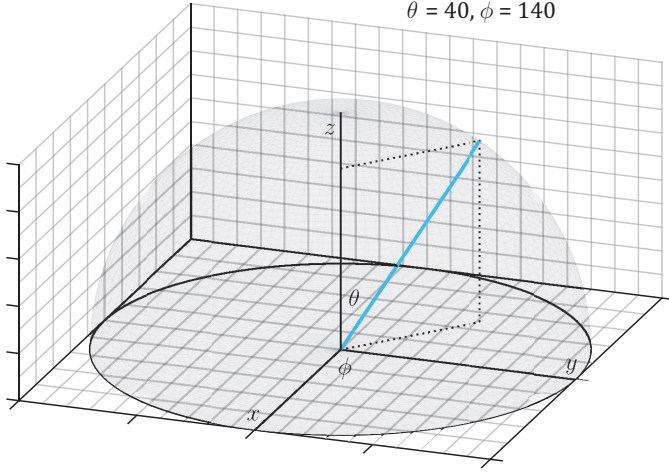


Fig. 6. Spherical coordinate system shown with elevation $\theta = 40$, and azimuth $\phi = 140$.

First consider the input to a single sensor as

$$y_m(t) = e^{j(\omega^\circ t - \vec{k}^\circ \cdot \vec{x}_m)} \quad (12)$$

where $\omega = 2\pi f$ is the frequency of the input signal with frequency f . The wavenumber vector (or wave vector) $\vec{k} = [k_x, k_y, k_z]$ is the propagation vector giving both the magnitude and direction of arrival of the incident plane wave. The $^\circ$ over \vec{k}° and ω° is to denote that the wave has a specific frequency ω° , and a specific direction given by the wave vector \vec{k}° , which may be different from the direction \vec{k} which the array is steered to. As before $\vec{x}_m = [x_m, y_m, z_m]$ is the position in space of the receiving sensor. By using the same input signal as in (12), the delayed signal may be stated as

$$\begin{aligned} y_m(t - \Delta_m) &= e^{j(\omega^\circ(t - \Delta_m) - \vec{k}^\circ \cdot \vec{x}_m)} \\ &= e^{j(\omega^\circ t - \vec{k}^\circ \cdot \vec{x}_m)} \cdot e^{-j\omega^\circ \Delta_m} \\ &= y_m(t) \cdot e^{-j\omega^\circ \Delta_m} \end{aligned} \quad (13)$$

Remember now that the signal $y_m(t)$ is the received signal from individual sensors and will be different for different sensors, as seen on the top left of Fig. 1, and $e^{-j\omega^\circ \Delta_m}$ represents the phase delay associated with the signal at the m th microphone. The DAS beamformer output may again be stated as in (1) as

$$z(t) = \sum_{m=0}^{M-1} w_m \cdot y_m(t) \cdot e^{-j\omega^\circ \Delta_m} \quad (14)$$

Now we want to choose the set of delays as used on the top right of Fig. 1 such that the phase shifts steer the beam's

listening direction to the direction of the vector \vec{k} which can be different from the waves propagation direction \vec{k}° . That is, the delays are chosen as

$$\Delta_m = -\frac{\vec{k}}{\omega^\circ} \cdot \vec{x}_m \quad (15)$$

and the total response from (14) may be calculated as

$$\begin{aligned} z(t) &= \sum_{m=0}^{M-1} w_m \cdot y_m(t) \cdot e^{-j\omega^\circ(-\frac{\vec{k}}{\omega^\circ} \cdot \vec{x}_m)} \\ &= \sum_{m=0}^{M-1} w_m \cdot y_m(t) \cdot e^{j\vec{k} \cdot \vec{x}_m} \end{aligned} \quad (16)$$

where $e^{j\vec{k} \cdot \vec{x}_m}$ is the phase delay associated with each individual sensor. Now to characterise the output of the DAS beamformer further, we write $y_m(t)$ as in (12) and insert it into (16).

$$\begin{aligned} z(t) &= \sum_{m=0}^{M-1} w_m \cdot y_m(t) \cdot e^{j\vec{k} \cdot \vec{x}_m} \\ &= \sum_{m=0}^{M-1} w_m \cdot e^{j(\omega^\circ t - \vec{k}^\circ \cdot \vec{x}_m)} \cdot e^{j\vec{k} \cdot \vec{x}_m} \\ &= \sum_{m=0}^{M-1} w_m e^{j(\vec{k} - \vec{k}^\circ) \cdot \vec{x}_m} \cdot e^{j\omega^\circ t} \\ &= W(\vec{k} - \vec{k}^\circ) \cdot e^{j\omega^\circ t} \end{aligned} \quad (17)$$

where

$$W(\vec{k}) = \sum_{m=0}^{M-1} w_m e^{j\vec{k} \cdot \vec{x}_m} \quad (18)$$

is the so called *array pattern* or *array factor* which is a function of the position of the sensors in the array and the weights used. In the case of uniform shading where the weights are all equal, the array pattern depends only on the array geometry. The function $W(\vec{k} - \vec{k}^\circ)$ given in (17) is called the *beampattern* of the array. We see how the beampattern describes how a monochromatic signal $e^{j\omega^\circ t}$ propagating in a direction given by \vec{k}° with a frequency ω° is attenuated by a DAS beamformer steered towards the direction \vec{k} . The beampattern will have maximum output when the steering direction coincides with the wave's direction of propagation, that is we set $\vec{k} = \vec{k}^\circ$.

Returning to the notation given in (16), if we now include the phase delays in the received signal vector $\mathbf{Y} = y_m(t) \cdot e^{j\vec{k} \cdot \vec{x}_m}$, we may write (16) in vector notation as

$$z(t) = \sum_{m=0}^{M-1} w_m \cdot (y_m(t) \cdot e^{j\vec{k} \cdot \vec{x}_m}) = \mathbf{w}^H \mathbf{Y} \quad (19)$$

where \mathbf{Y} is the $M \times 1$ vector of the received signal from each sensor with its associated phase delay



$$\mathbf{Y} = \begin{bmatrix} y_0(t) \cdot e^{j\vec{k}\cdot\vec{x}_0} \\ y_1(t) \cdot e^{j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ y_{M-1}(t) \cdot e^{j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix} \quad (20)$$

\mathbf{w} is the $M \times 1$ vector of weights for individual sensors

$$\mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{M-1} \end{bmatrix} \quad (21)$$

and H denotes the complex conjugate transpose. By using the vector notation given in (19), and assuming we have already steered the array to the desired direction, we can calculate the power, or the variance, of the output signal as

$$\begin{aligned} P(z(t)) &= \sigma^2 = \mathbb{E}\{|z(t)|^2\} \\ &= \mathbb{E}\{(\mathbf{w}^H \mathbf{Y})(\mathbf{w}^H \mathbf{Y})^H\} \\ &= \mathbb{E}\{\mathbf{w}^H \mathbf{Y} \mathbf{Y}^H \mathbf{w}\} \\ &= \mathbf{w}^H \mathbb{E}\{\mathbf{Y} \mathbf{Y}^H\} \mathbf{w} \\ &= \mathbf{w}^H \mathbf{R} \mathbf{w} \end{aligned} \quad (22)$$

The above expression gives the power of the beamformer's output in the steered direction, where $\mathbf{R} = \mathbb{E}\{\mathbf{Y} \mathbf{Y}^H\}$ is the correlation matrix of the data. Now suppose we want to measure the output power as a function of steering directions, or scanning angles. In (16) the phase delays associated with each individual sensor $e^{j\vec{k}\cdot\vec{x}_m}$ is the so called *steering vector*, denoted as \mathbf{e} , and governs how we want to steer the beam of our array

$$\mathbf{e} = e^{j\vec{k}\cdot\vec{x}_m} = \begin{bmatrix} e^{j\vec{k}\cdot\vec{x}_0} \\ e^{j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix} \quad (23)$$

For a wave propagating in spherical coordinates, the wave vector is related to the Cartesian coordinates by simple trigonometric formulas

$$\begin{aligned} k_x &= k \sin \theta \cos \phi \\ k_y &= k \sin \theta \sin \phi \\ k_z &= k \cos \theta \end{aligned} \quad (24)$$

where the x -component of the wave vector, k_x , determines the rate of change of the phase of a propagating plane wave in the x -direction. The same definitions apply for the y - and z -directions. The wavenumber k is equal to $2\pi/\lambda$ or $2\pi c/f$. The steering vector then depends on the frequency and propagation direction of the incoming plane wave, and can be expressed in terms of wavelength λ , elevation θ and azimuth ϕ . Usually planar 2D arrays with the elements

positioned in the same plane will be used, so that the z -coordinate of the sensors will be equal to zero. This means that the dependence on z and k_z may be omitted, and the steering vector can be written as

$$\mathbf{e} = e^{j\vec{k}\cdot\vec{x}_m} = \begin{bmatrix} e^{j\frac{2\pi}{\lambda}(\sin \theta \cos \phi \cdot x_0 + \sin \theta \sin \phi \cdot y_0)} \\ e^{j\frac{2\pi}{\lambda}(\sin \theta \cos \phi \cdot x_1 + \sin \theta \sin \phi \cdot y_1)} \\ \vdots \\ e^{j\frac{2\pi}{\lambda}(\sin \theta \cos \phi \cdot x_{M-1} + \sin \theta \sin \phi \cdot y_{M-1})} \end{bmatrix} \quad (25)$$

In (22) we already assumed the array was steered to the correct direction before calculating the power. If we now want to calculate the energy for an arbitrary direction instead, we must realize that since the received signal vector \mathbf{Y} in (22) have phase delays included, this must mean that \mathbf{R} also is a function of the steering vector \mathbf{e} , that is $\mathbf{R}(\mathbf{e}) = \mathbf{e}^H \mathbf{R} \mathbf{e}$. Now suppose we want to measure the output power as a function of scanning angle, or rather as a function of steering vector. Calculating the output power as a function of steering vector is termed the *steered response* and is the power of the beamformer output in the frequency domain. This array output power spectral density may then be expressed by using the correlation matrix and the steering vector as

$$\begin{aligned} P(\mathbf{e}) &= \mathbf{w}^H \mathbf{R}(\mathbf{e}) \mathbf{w} \\ &= \mathbf{w}^H (\mathbf{e}^H \mathbf{R} \mathbf{e}) \mathbf{w} \end{aligned} \quad (26)$$

For a uniform array where all sensors have equal weight, the above expression reduces to

$$P(\mathbf{e}) = \mathbf{e}^H \mathbf{R} \mathbf{e} \quad (27)$$

In essence to calculate the spatial spectrum of the DAS beamformer for a specific array, use (26) to calculate the output power, or (27) for a uniformly weighted array. The calculation will be performed for each steering vector, where each steering vector corresponds to exactly one pair of θ, ϕ scanning angles.

